M. Math 2008-2009 Advanced Functional Analysis, Final examination

05-12-2008 Time: 3hrs

Show complete work. You may assume standard results in Banach space theory, but need to quote them correctly. Each question is worth 6 points.

1) Let $X$ be a metrizable TVS and Y a TVS. Let $T : X \to Y$ be a linear map such that for every sequence $\{x_n\}_{n\geq 1} \subset X$, $x_n \to 0$ implies, $\{T(x_n)\}_{n\geq 1}$ is a bounded sequence. Show that $T$ is continuous.

2) Consider $\Lambda_m : \ell^2 \to C$ defined by $\Lambda_m(x) = \sum_1^m n^2 x(n)$. Let $x_n = \frac{1}{n} e_n$. Show that $K = \{x_n\}_{n\geq 1} \cup \{0\}$ is compact. Show that each $\Lambda_m(K)$ is a bounded but $\{\Lambda_m(K)\}_{m\geq 1}$ is not uniformly bounded.

3) Let $X$ be a separable normed linear space. Show that $X^*$, with the weak*-topology is separable.

4) Let $X$ be a LCTVS and $A, B$ two compact convex sets. Show that the extreme points of the convex hull, $\partial_e CO(A \cup B) = \partial_e A \cup \partial_e B$ .

5) Let $K$ be a compact convex set in a LCTVS $X$. Let $a : K \to [0,1]$ be an onto affine continuous map. Show that $a^{-1}(0)$ is an extreme convex set.

6) Let $K$ be a compact convex set in a LCVTS space. Let $x_0 \in K$. Show that there exists a probability measure $\mu$ with $\mu((\partial_e K)^-) = 1$ and the resultant , $\gamma(\mu) = x_0$.

7) Let $([0,1], \mathcal{A}, \lambda)$ be the Lebesgue space. Let $f : [0,1] \to (C([0,1])_1, weak)$ be a measurable function (subscript 1 denotes the closed unit ball). If for every Lebesgue measurable set $E$, $\int_E f \, d\lambda = 0$. Show that $f = 0$ a.e.

8) Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Let $X$ be a Banach space. Let $G : \mathcal{A} \to X$ be a finitely additive measure. Suppose $||G(E)|| \leq \mu(E)$ for all $E \in \mathcal{A}$. Show that $G$ is of bounded variation.

9) Give complete details to show that any closed subspace of $\ell^1$ has the Radon-Nikodym property.

10) State and prove the Exhaustion Lemma for vector measures.